

A screw dislocation in a functionally graded material using the translation gauge theory of dislocations

Markus Lazar^{a,b,*}

^a Heisenberg Research Group,
Department of Physics,
Darmstadt University of Technology,
Hochschulstr. 6,
D-64289 Darmstadt, Germany

^b Department of Physics,
Michigan Technological University,
Houghton, MI 49931, USA

January 12, 2013

Abstract

The aim of this paper is to provide new results and insights for a screw dislocation in functionally graded media within the gauge theory of dislocations. We present the equations of motion for dislocations in inhomogeneous media. We specify the equations of motion for a screw dislocation in a functionally graded material. The material properties are assumed to vary exponentially along the x and y -directions. In the present work we give the analytical gauge field theoretic solution to the problem of a screw dislocation in inhomogeneous media. Using the dislocation gauge approach, rigorous analytical expressions for the elastic distortions, the force stresses, the dislocation density and the pseudomoment stresses are obtained depending on the moduli of gradation and an effective intrinsic length scale characteristic for the functionally graded material under consideration.

Keywords: Screw dislocation; Functionally graded material; Dislocation gauge theory; Size-effects.

**E-mail address:* lazar@fkp.tu-darmstadt.de (M. Lazar).

1 Introduction

Nonhomogeneous media, multilayered structures and functionally graded materials (FGMs) are of considerable technical and engineering importance as well as of mathematical interest (see, e.g., Erdogan (1995)). Generally, such materials refer to heterogenous composite materials in which the material moduli vary smoothly and continuously from point to point. Typical examples of FGMs are ceramic/ceramic and metal/ceramic systems. Comprehensive reviews on aspects of FGMs can be found in Markworth et al. (1995) and Erdogan (1995). However, only few investigations have been made to assess the role and the importance of dislocations in FGMs. For the first time, Barnett (1972) found the displacement and stress fields of a screw dislocation in an isotropic medium which is arbitrarily graded in the x -direction. One important example for a FGM are exponentially graded materials. For such media the elasticity moduli are exponential functions depending on the space coordinates and the new material parameters describing the gradation (see, e.g., Erdogan (1995)). The Green function for a two dimensional exponentially graded elastic medium has been found by Chan et al. (2004) and the three-dimensional Green function is given by Martin et al. (2002).

Classical continuum theories possess no intrinsic length scales and, therefore, they are scale-free continuum theories which are not able to describe size effects being of importance at micro- and nanoscales and near defects. In order to describe such effects generalized continuum theories are needed. Such generalized continuum theories are, for instance, strain gradient theories (Mindlin, 1964; Altan and Aifantis, 1997; Lazar and Maugin, 2005) and the dislocation gauge theory (Kadić and Edelen, 1983; Edelen and Lagoudas, 1988; Lazar, 2002; Lazar and Anastassiadis, 2009) which enrich the classical theories with additional material lengths. Using strain gradient elasticity, cracks in FGMs have been investigated by Paulino et al. (2003) and Chan et al. (2008). For the first time ever, the solution of a screw dislocation in FGMs, using the theory of gradient elasticity, was given by Lazar (2007). Gradient elasticity is a theory which is very popular in engineering sciences. But not everything is well-understood about the physical importance of the higher-order stresses (hyperstresses). Gradient elasticity should be understood as an effective theory since it was invented for compatible deformations. On the other hand, a more physically motivated ‘gradient-like’ theory is the so-called dislocation gauge theory. In the dislocation gauge theory both the incompatible elastic distortion tensor and the dislocation density tensor are physical state quantities giving contribution to the elastic stored energy density. Also for nonhomogeneous media these physical state quantities are gauge-invariant. In this approach, the higher-order stress is realized as pseudomoment stress which is the response to the presence of dislocations. This pseudomoment stress is related to the moment stress known from generalized elasticities like Cosserat elasticity (see, e.g., Nowacki (1986); Eringen (1999)). Thus, the force stress tensor is asymmetric like in Cosserat elasticity. In general, in gradient elasticity the hyperstress (double stress) possesses five material moduli (Mindlin, 1964). In a simplified version of gradient elasticity (Altan and Aifantis, 1997) under ad-hoc assumptions the double stress can be simplified. In the dislocation gauge theory such assumptions are not necessary. For that reason we use the gauge theory of dislocations in this paper.

In this paper we study a screw dislocation in FGM in the framework of the translation

gauge theory of dislocations. In Section 2 we start with the general framework of the translational gauge theory of dislocations and we derive the isotropic three-dimensional field equations of dislocations for nonhomogeneous media as well as for exponentially graded materials. In Section 3 we specialize in the anti-plane problem and we give the gauge-theoretical solution of a screw dislocation in exponentially graded materials. We find rigorous and ‘simple’ analytical solutions for the force stresses, elastic distortion, torsion and pseudomoment stresses produced by a screw dislocation in FGMs. Our hope is that the gauge solution will provide new physical insight to possible improvements of designing FGMs.

2 Gauge theory of dislocations

This section introduces the notation and constitutive equations of the translational gauge theory of dislocations (Lazar and Anastassiadis, 2009), which will be used to investigate a screw dislocation in FGMs. In the dislocation gauge theory, the elastic distortion tensor β_{ij} is incompatible due to the occurrence of a so-called translational gauge field ϕ_{ij}

$$\beta_{ij} = u_{i,j} + \phi_{ij}, \quad (1)$$

where u_i is the displacement vector. It is important to note that u_i and ϕ_{ij} are just the canonical field quantities and they are not unique and not physical state quantities. In presence of dislocations, u_i and ϕ_{ij} are discontinuous (or multivalued) fields. The gauge field ϕ_{ij} may be identified with the negative plastic distortion. The appearance of incompatibility in form of the translation gauge field gives rise to an incompatibility tensor in the framework of the translation gauge theory of dislocations which is called the dislocation density tensor or torsion tensor. Accordingly, the dislocation density tensor is given in terms of the gauge field

$$T_{ijk} = \phi_{ik,j} - \phi_{ij,k}, \quad T_{ijk} = -T_{ikj} \quad (2)$$

and, alternatively, in terms of the elastic distortion

$$T_{ijk} = \beta_{ik,j} - \beta_{ij,k}. \quad (3)$$

In the present framework, the elastic distortion tensor β_{ij} and the dislocation density tensor T_{ijk} are the physical state quantities. Due to the form of the dislocation density tensor (2) and (3) it fulfills the translational Bianchi identity

$$\epsilon_{jkl} T_{ijk,l} = 0. \quad (4)$$

Eq. (4) is the well-known conservation law of dislocations and states that a dislocation line cannot end inside the body.

In the static case of the (linear) translation gauge theory of dislocations, the Lagrangian density is of the bilinear form

$$\mathcal{L} = -W = -\frac{1}{2} \sigma_{ij} \beta_{ij} - \frac{1}{4} H_{ijk} T_{ijk}, \quad (5)$$

where W denotes the stored energy density. The canonical conjugate quantities (response quantities) are defined by

$$\sigma_{ij} := -\frac{\partial \mathcal{L}}{\partial \beta_{ij}}, \quad H_{ijk} := -2\frac{\partial \mathcal{L}}{\partial T_{ijk}}, \quad H_{ijk} = -H_{ikj}, \quad (6)$$

where σ_{ij} and H_{ijk} are the force stress tensor and the pseudomoment stress tensor, respectively. Here, the force stress tensor σ_{ij} is asymmetric. The moment stress tensor $\tau_{ijk} = -\tau_{jik}$ can be obtained from the pseudomoment stress tensor: $\tau_{ijk} = -H_{[ij]k}$ (see Lazar and Anastassiadis (2009); Lazar and Hehl (2010)). They have the dimensions: $[\sigma_{ij}] = \text{force}/(\text{length})^2 \stackrel{\text{SI}}{=} \text{Pa}$ and $[H_{ijk}] = \text{force}/\text{length} \stackrel{\text{SI}}{=} \text{N/m}$.

The Euler-Lagrange equations with respect to the canonical field variables derived from the total Lagrangian density $\mathcal{L} = \mathcal{L}(\beta_{ij}, T_{ijk})$ are given by

$$E_i^u(\mathcal{L}) = \partial_j \frac{\partial \mathcal{L}}{\partial u_{i,j}} - \frac{\partial \mathcal{L}}{\partial u_i} = 0, \quad (7)$$

$$E_{ij}^\phi(\mathcal{L}) = \partial_k \frac{\partial \mathcal{L}}{\partial \phi_{ij,k}} - \frac{\partial \mathcal{L}}{\partial \phi_{ij}} = 0. \quad (8)$$

We may add to \mathcal{L} a so-called null Lagrangian, $\mathcal{L}_N = \sigma_{ij}^0 \beta_{ij}$, with the ‘background’ stress σ_{ij}^0 satisfying: $\sigma_{ij,j}^0 = 0$. In terms of the canonical conjugate quantities (6), Eqs. (7) and (8) take the form

$$\sigma_{ij,j} = 0, \quad (\text{force balance of elasticity}), \quad (9)$$

$$H_{ijk,k} + \sigma_{ij} = \sigma_{ij}^0, \quad (\text{stress balance of dislocations}). \quad (10)$$

The force stresses are the sources of the pseudomoment stress. In fact, one can see in Eq. (10) that the source of the pseudomoment stress tensor is an effective force stress tensor $(\sigma_{ij} - \sigma_{ij}^0)$ (see, e.g., Edelen and Lagoudas (1988)). This effective stress tensor, which is the difference between the gauge-theoretical stress field and the background stress field, drives the dislocation fields.

The linear, isotropic constitutive relations for the force stress and the pseudomoment stress are

$$\sigma_{ij} = \lambda \delta_{ij} \beta_{kk} + \mu (\beta_{ij} + \beta_{ji}) + \gamma (\beta_{ij} - \beta_{ji}), \quad (11)$$

$$H_{ijk} = c_1 T_{ijk} + c_2 (T_{jki} - T_{kji}) + c_3 (\delta_{ij} T_{llk} - \delta_{ik} T_{llj}). \quad (12)$$

Here μ, λ, γ are the elastic stiffness moduli and c_1, c_2, c_3 denote the resistivity moduli associated with dislocations. The six material moduli have the dimensions: $[\mu, \lambda, \gamma] = \text{force}/(\text{length})^2 \stackrel{\text{SI}}{=} \text{Pa}$ and $[c_1, c_2, c_3] = \text{force} \stackrel{\text{SI}}{=} \text{N}$. Here, the material moduli are nonhomogeneous that means they depend on the space coordinates: $\mu \equiv \mu(\mathbf{x})$, $\lambda \equiv \lambda(\mathbf{x})$, $\gamma \equiv \gamma(\mathbf{x})$, $c_1 \equiv c_1(\mathbf{x})$, $c_2 \equiv c_2(\mathbf{x})$ and $c_3 \equiv c_3(\mathbf{x})$. The constitutive relations for nonhomogeneous media (11) and (12) have the same formal form as the relations for homogeneous media. The only difference is that the constitutive moduli are not constant.

The requirement of non-negativity of the stored energy (material stability) $W \geq 0$ leads to the conditions of semi-positiveness of the constitutive moduli. Particularly, the

constitutive moduli have to fulfill the following conditions (Lazar and Anastassiadis, 2009)

$$\mu \geq 0, \quad \gamma \geq 0, \quad 3\lambda + 2\mu \geq 0, \quad (13)$$

$$c_1 - c_2 \geq 0, \quad c_1 + 2c_2 \geq 0, \quad c_1 - c_2 + 2c_3 \geq 0. \quad (14)$$

If we substitute the constitutive equations (11) and (12) into the Euler-Lagrange equations (9) and (10) and use the definition (3), we obtain the explicite form of the Euler-Lagrange equations for a nonhomogeneous material

$$\lambda \beta_{jj,i} + (\mu + \gamma) \beta_{ij,j} + (\mu - \gamma) \beta_{ji,j} + \lambda_{,i} \beta_{jj} + (\mu + \gamma)_{,j} \beta_{ij} + (\mu - \gamma)_{,j} \beta_{ji} = 0, \quad (15)$$

$$\begin{aligned} c_1(\beta_{ik,jk} - \beta_{ij,kk}) + c_2(\beta_{ji,kk} - \beta_{jk,ik} + \beta_{kj,ik} - \beta_{ki,jk}) + c_3[\delta_{ij}(\beta_{lk,lk} - \beta_{ll,kk}) + \beta_{kk,ji} - \beta_{kj,ki}] \\ + c_{1,k}(\beta_{ik,j} - \beta_{ij,k}) + c_{2,k}(\beta_{ji,k} - \beta_{jk,i} + \beta_{kj,i} - \beta_{ki,j}) + c_{3,k} \delta_{ij}(\beta_{lk,l} - \beta_{ll,k}) + c_{3,i}(\beta_{kk,j} - \beta_{kj,k}) \\ + \lambda \delta_{ij}(\beta_{kk} - \beta_{kk}^0) + (\mu + \gamma)(\beta_{ij} - \beta_{ij}^0) + (\mu - \gamma)(\beta_{ji} - \beta_{ji}^0) = 0, \end{aligned} \quad (16)$$

where β_{ij}^0 denotes the ‘classical’ elastic distortion tensor. It can be seen that the space-dependence of the material moduli influences the field equations (15) and (16) due to gradients of the material moduli. Eq. (15) is a partial differential equation of first order for the elastic distortion as well as for the constitutive moduli. On the other hand, Eq. (16) contains differential operators of second order acting on the elastic distortion and ‘mixed’ gradients of the constitutive moduli and the elastic distortions.

Rather than looking for the general inhomogeneous behaviour of the constitutive moduli, we specify here on an inhomogeneous material, which can find application to functionally graded materials. We assume that the material properties vary in a simple, explicite manner. Here, we consider exponential variations of the six constitutive moduli in the following manner

$$\begin{aligned} \lambda = \lambda^0 e^{2(a_1x+a_2y+a_3z)}, \quad \mu = \mu^0 e^{2(a_1x+a_2y+a_3z)}, \quad \gamma = \gamma^0 e^{2(a_1x+a_2y+a_3z)}, \\ c_1 = c_1^0 e^{2(a_1x+a_2y+a_3z)}, \quad c_2 = c_2^0 e^{2(a_1x+a_2y+a_3z)}, \quad c_3 = c_3^0 e^{2(a_1x+a_2y+a_3z)}, \end{aligned} \quad (17)$$

where $\lambda^0, \mu^0, \gamma^0, c_1^0, c_2^0, c_3^0, a_1, a_2$ and a_3 are the material parameters of the FGM. The moduli a_1, a_2 and a_3 are the characteristic parameters for exponential gradation in x, y and z -direction and they have the dimensions: $[a_1, a_2, a_3] = 1/\text{length}$. The parameters a_1, a_2 and a_3 determine the gradation of all six constitutive moduli $\lambda, \mu, \gamma, c_1, c_2, c_3$. For such an exponentially FGM, the equations (15) and (16) reduce to

$$\lambda(\nabla_i + 2a_i)\beta_{jj} + (\mu + \gamma)(\nabla_j + 2a_j)\beta_{ij} + (\mu - \gamma)(\nabla_j + 2a_j)\beta_{ji} = 0, \quad (18)$$

$$\begin{aligned} c_1(\nabla_k + 2a_k)(\nabla_j \beta_{ik} - \nabla_k \beta_{ij}) + c_2(\nabla_k + 2a_k)(\nabla_k \beta_{ji} - \nabla_i \beta_{jk} + \nabla_i \beta_{kj} - \nabla_j \beta_{ki}) \\ + c_3 \delta_{ij}(\nabla_k + 2a_k)(\nabla_l \beta_{lk} - \nabla_k \beta_{ll}) + c_3(\nabla_i + 2a_i)(\nabla_j \beta_{kk} - \nabla_k \beta_{kj}) \\ + \lambda \delta_{ij}(\beta_{kk} - \beta_{kk}^0) + (\mu + \gamma)(\beta_{ij} - \beta_{ij}^0) + (\mu - \gamma)(\beta_{ji} - \beta_{ji}^0) = 0, \end{aligned} \quad (19)$$

where ∇_i denotes the three-dimensional Nabla operator. Eqs. (18) and (19) are the three-dimensional governing equations of dislocations in exponentially graded materials using the dislocation gauge theory.

3 Gauge solution of a screw dislocation in functionally graded materials

We now derive the equations of motion for a screw dislocation in FGMs. We consider an infinitely long screw dislocation parallel to the z -axis with the Burgers vector $b = b_z$. The symmetry of such a straight screw dislocation leaves only the following non-vanishing components of the elastic distortion tensor (see, e.g., deWit (1973)): β_{zx} , β_{zy} , and for the dislocation density tensor: T_{zxy} .

For the anti-plane problem of a screw dislocation in nonhomogeneous materials, the equilibrium condition (15) reduces to

$$\beta_{zj,j} = -\frac{(\mu + \gamma)_{,j}}{\mu + \gamma} \beta_{zj}, \quad j = x, y. \quad (20)$$

Using equation (20), we obtain from (16) the following equations for the components β_{zx} and β_{zy} of a screw dislocation in nonhomogeneous materials

$$c_1 \beta_{zx,jj} + c_{1,j} \beta_{zx,j} - \left(c_{1,j} - c_1 \frac{(\mu + \gamma)_{,j}}{\mu + \gamma} \right) \beta_{zj,x} - (\mu + \gamma) (\beta_{zx} - \beta_{zx}^0) = 0, \quad (21)$$

$$c_2 \beta_{zx,jj} + c_{2,j} \beta_{zx,j} - \left(c_{2,j} - c_2 \frac{(\mu + \gamma)_{,j}}{\mu + \gamma} \right) \beta_{zj,x} + (\mu - \gamma) (\beta_{zx} - \beta_{zx}^0) = 0, \quad (22)$$

$$c_1 \beta_{zy,jj} + c_{1,j} \beta_{zy,j} - \left(c_{1,j} - c_1 \frac{(\mu + \gamma)_{,j}}{\mu + \gamma} \right) \beta_{zj,y} - (\mu + \gamma) (\beta_{zy} - \beta_{zy}^0) = 0, \quad (23)$$

$$c_2 \beta_{zy,jj} + c_{2,j} \beta_{zy,j} - \left(c_{2,j} - c_2 \frac{(\mu + \gamma)_{,j}}{\mu + \gamma} \right) \beta_{zj,y} + (\mu - \gamma) (\beta_{zy} - \beta_{zy}^0) = 0. \quad (24)$$

For the two-dimensional anti-plane problem of FGMs, we assume that the constitutive moduli are exponentially graded,

$$\mu = \mu^0 e^{2(a_1 x + a_2 y)}, \quad \gamma = \gamma^0 e^{2(a_1 x + a_2 y)}, \quad c_1 = c_1^0 e^{2(a_1 x + a_2 y)}, \quad c_2 = c_2^0 e^{2(a_1 x + a_2 y)}, \quad (25)$$

where μ^0 , γ^0 , c_1^0 , c_2^0 , a_1 and a_2 are the material constants of the anti-plane problem. The moduli a_1 and a_2 are the characteristic parameters of exponential gradation in x and y -direction, respectively. If we substitute (25) into (21)–(24), the governing equations are

$$c_1 (\Delta + 2\mathbf{a} \cdot \nabla) \beta_{zx} - (\mu + \gamma) (\beta_{zx} - \beta_{zx}^0) = 0, \quad (26)$$

$$c_2 (\Delta + 2\mathbf{a} \cdot \nabla) \beta_{zx} + (\mu - \gamma) (\beta_{zx} - \beta_{zx}^0) = 0, \quad (27)$$

$$c_1 (\Delta + 2\mathbf{a} \cdot \nabla) \beta_{zy} - (\mu + \gamma) (\beta_{zy} - \beta_{zy}^0) = 0, \quad (28)$$

$$c_2 (\Delta + 2\mathbf{a} \cdot \nabla) \beta_{zy} + (\mu - \gamma) (\beta_{zy} - \beta_{zy}^0) = 0, \quad (29)$$

where Δ and ∇ denote the two-dimensional Laplacian and the Nabla operator, respectively. It can be seen in (26)–(29) that we have four equations for only two components β_{zx} and β_{zy} . Thus, it is clear that not all four equations (26)–(29) are independent. In fact, if we compare (26) with (27) as well as (28) with (29), we find the relation between the material moduli

$$\frac{c_1}{\mu + \gamma} = -\frac{c_2}{\mu - \gamma}. \quad (30)$$

In addition, we may define the following intrinsic gauge length of the anti-plane problem of a screw dislocation

$$\ell = \sqrt{\frac{c_1}{\mu + \gamma}} = \sqrt{\frac{c_1^0}{(\mu + \gamma)^0}}. \quad (31)$$

Evidently, the length scale ℓ is constant for FGMs using (25). This fact is in agreement with gradient elasticity for FGMs (Paulino et al., 2003; Chan et al., 2006, 2008; Lazar, 2007) where the characteristic gradient length is also constant. Only the constitutive moduli entering the constitutive relations are nonconstant.

By means of (30) and (31) the material moduli c_1 and c_2 can be expressed in terms of the length ℓ , and the constitutive moduli μ and γ

$$c_1 = \ell^2(\mu + \gamma), \quad c_2 = -\ell^2(\mu - \gamma). \quad (32)$$

Thus, the gradation of c_1 and c_2 is given by the gradation of the constitutive moduli μ and γ . Accordingly, the governing equations for β_{zx} and β_{zy} reduce to

$$[1 - \ell^2(\Delta + 2\mathbf{a} \cdot \nabla)]\beta_{zx} = \beta_{zx}^0, \quad (33)$$

$$[1 - \ell^2(\Delta + 2\mathbf{a} \cdot \nabla)]\beta_{zy} = \beta_{zy}^0. \quad (34)$$

Due to the gradation term $\mathbf{a} \cdot \nabla$, Eqs. (33) and (34) may be called ‘perturbed’ Helmholtz equations.

Differentiating equation (33) with respect to y and equation (34) with respect to x and using the definition of the torsion (3), the governing equation for the torsion (dislocation density) of a screw dislocation in FGM turns out to be

$$[1 - \ell^2(\Delta + 2\mathbf{a} \cdot \nabla)]T_{zxy} = T_{zxy}^0, \quad (35)$$

where

$$T_{zxy}^0 = b\delta(x)\delta(y) \quad (36)$$

is the dislocation density of a straight screw dislocation in classical elasticity. Using the substitution

$$T_{zxy} = e^{-(a_1x+a_2y)}\Psi, \quad (37)$$

we obtain from Eq. (35) the following Helmholtz equation

$$[\Delta - \kappa^2]\Psi = -\frac{b}{\ell^2}\delta(x)\delta(y), \quad (38)$$

with

$$\kappa = \sqrt{\frac{1 + a^2\ell^2}{\ell^2}}, \quad a = \sqrt{a_1^2 + a_2^2}. \quad (39)$$

It can be seen that κ is the inverse intrinsic length characteristic for the anti-plane problem of FGMs in the dislocation gauge approach. Therefore, $1/\kappa$ is an effective length scale

given in terms of the gauge length scale ℓ and the gradation length scale $1/a$. The inverse length scale of gradation a is also constant. Thus, in addition to $1/a$, two (constant) characteristic length scales, namely ℓ and $1/\kappa$, appear in the gauge-theoretical approach of the anti-plane problem of FGMs. The solution of (38) is given by

$$\Psi = \frac{b}{2\pi\ell^2} K_0(\kappa r), \quad (40)$$

where $r = \sqrt{x^2 + y^2}$ and K_n is the n th order modified Bessel function of the second kind. Finally, the torsion of a screw dislocation in FGM using the dislocation gauge theory is obtained as

$$T_{zxy} = \frac{b}{2\pi\ell^2} e^{-(a_1x+a_2y)} K_0(\kappa r). \quad (41)$$

If we compare the dislocation density (41) with the corresponding one for homogeneous media (see, e.g., Lazar and Anastassiadis (2009)), we observe the exponential factor and the length scale κ instead of $1/\ell$ as difference. That means the dislocation density (41) has lost the cylindrical symmetry and decays faster in the far-field in FGMs. Substituting (41) into the constitutive relation (12) and making use of (30), we obtain for the components of the pseudomoment stress tensor

$$H_{zxy} = \frac{(\mu + \gamma)^0 b}{2\pi} e^{(a_1x+a_2y)} K_0(\kappa r), \quad (42)$$

$$H_{xyz} = -\frac{(\mu - \gamma)^0 b}{2\pi} e^{(a_1x+a_2y)} K_0(\kappa r), \quad (43)$$

$$H_{yzx} = -\frac{(\mu - \gamma)^0 b}{2\pi} e^{(a_1x+a_2y)} K_0(\kappa r). \quad (44)$$

Due to the gradation, the fields (41)–(44) do not possess cylindrical symmetry in contrast to the results for homogeneous media. It can be seen that the expressions (41)–(44) for $a_1 \rightarrow 0$ and $a_2 \rightarrow 0$ coincide with the result obtained by Lazar and Anastassiadis (2009) for homogeneous materials.

To solve the remaining equations (33) and (34), we use the constitutive relations (11) for the components σ_{zx} and σ_{zy} . In order to fulfill the equilibrium condition for the forces stresses $\sigma_{zx,x} + \sigma_{zy,y} = 0$, we introduce the stress function F according to

$$\sigma_{zx} = (\mu + \gamma)\beta_{zx} = -F_{,y}, \quad (45)$$

$$\sigma_{zy} = (\mu + \gamma)\beta_{zy} = F_{,x}. \quad (46)$$

By the help of equations (45) and (46), we are able to express the elastic distortion β_{zx} and β_{zy} in terms of the stress function F according

$$\beta_{zx} = -\frac{1}{\mu + \gamma} F_{,y}, \quad (47)$$

$$\beta_{zy} = \frac{1}{\mu + \gamma} F_{,x}. \quad (48)$$

Of course, Eqs. (47) and (48) satisfy the condition (20). If we substitute (47) and (48) into (33) and (34), we find the following partial differential equation for the stress function F

$$[1 - \ell^2(\Delta - 2\mathbf{a} \cdot \nabla)]F = F^0, \quad (49)$$

where F^0 on the right hand side satisfies the equation

$$[\Delta - 2\mathbf{a} \cdot \nabla]F^0 = (\mu + \gamma)^0 b \delta(x)\delta(y). \quad (50)$$

Eq. (50) is a ‘perturbed’ Laplace equation. The solution of (50), which is actually the Green function, reads (see Lazar (2007); Wang and Pan (2008))

$$F^0 = -\frac{(\mu + \gamma)^0 b}{2\pi} e^{(a_1 x + a_2 y)} K_0(ar). \quad (51)$$

In addition, we use the ansatz for F

$$F = F^0 + e^{(a_1 x + a_2 y)} \Phi \quad (52)$$

and obtain from (49) the Helmholtz equation for Φ

$$[\Delta - \kappa^2] \Phi = -(\mu + \gamma)^0 b \delta(x)\delta(y). \quad (53)$$

The solution of (53) is given by

$$\Phi = \frac{(\mu + \gamma)^0 b}{2\pi} K_0(\kappa r). \quad (54)$$

Eventually, the stress function F is calculated as

$$F = -\frac{(\mu + \gamma)^0 b}{2\pi} e^{(a_1 x + a_2 y)} [K_0(ar) - K_0(\kappa r)]. \quad (55)$$

Substituting (55) into (47) and (48), we arrive at the expressions of the elastic distortions as

$$\beta_{zx} = -\frac{b}{2\pi} e^{-(a_1 x + a_2 y)} \left(\frac{y}{r} [aK_1(ar) - \kappa K_1(\kappa r)] - a_2 [K_0(ar) - K_0(\kappa r)] \right), \quad (56)$$

$$\beta_{zy} = \frac{b}{2\pi} e^{-(a_1 x + a_2 y)} \left(\frac{x}{r} [aK_1(ar) - \kappa K_1(\kappa r)] - a_1 [K_0(ar) - K_0(\kappa r)] \right). \quad (57)$$

The elastic distortions β_{zx} and β_{zy} are plotted in Figs. (1) and (2). First of all, it can be seen that the elastic distortions (56) and (57) are nonsingular unlike the result for FGMs using elasticity theory and that they are given in terms of two inverse length scales a and κ . The component $\beta_{zx}(0, y)$ has extremum values near the dislocation line at $y = 0$ and it has a finite value at $y = 0$ depending on the material moduli. Unlike homogeneous media, it does not satisfy: $\beta_{zx}(0, y) \neq -\beta_{zx}(0, -y)$. Also $\beta_{zx}(x, 0)$ possesses a maximum at $x = 0$ and is asymmetric with respect to the plane $x = 0$: $\beta_{zx}(x, 0) \neq \beta_{zx}(-x, 0)$. Analogously, $\beta_{zy}(x, 0)$ has extremum values near the dislocation line at $x = 0$ and it has a finite value at $x = 0$. It does not fulfill, $\beta_{zy}(x, 0) \neq -\beta_{zy}(-x, 0)$, as for homogeneous

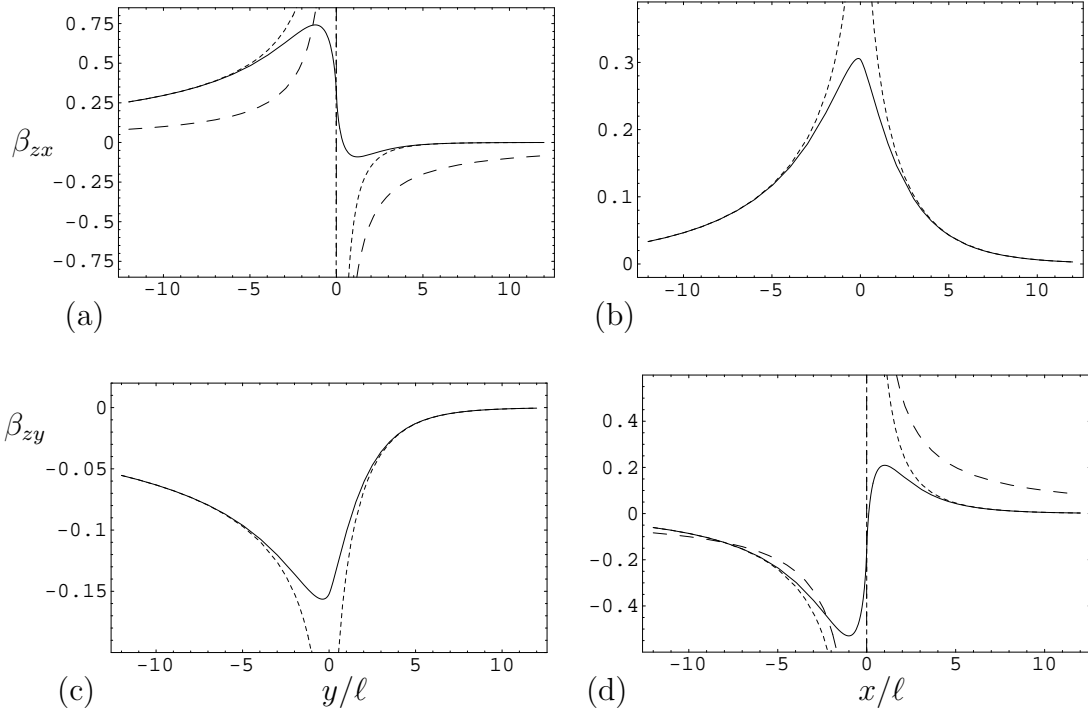


Figure 1: Elastic distortion of a screw dislocation in FGM using gauge theory (solid line) and elasticity (small dashed line) are given in units of $b/[2\pi]$ and with $a_1 = 0.1/\ell$ and $a_2 = 0.2/\ell$: (a) $\beta_{zx}(0, y)$, (b) $\beta_{zx}(x, 0)$, (c) $\beta_{zy}(0, y)$, (d) $\beta_{zy}(x, 0)$. The dashed curves represent the elastic distortion in classical elasticity of a homogeneous medium.

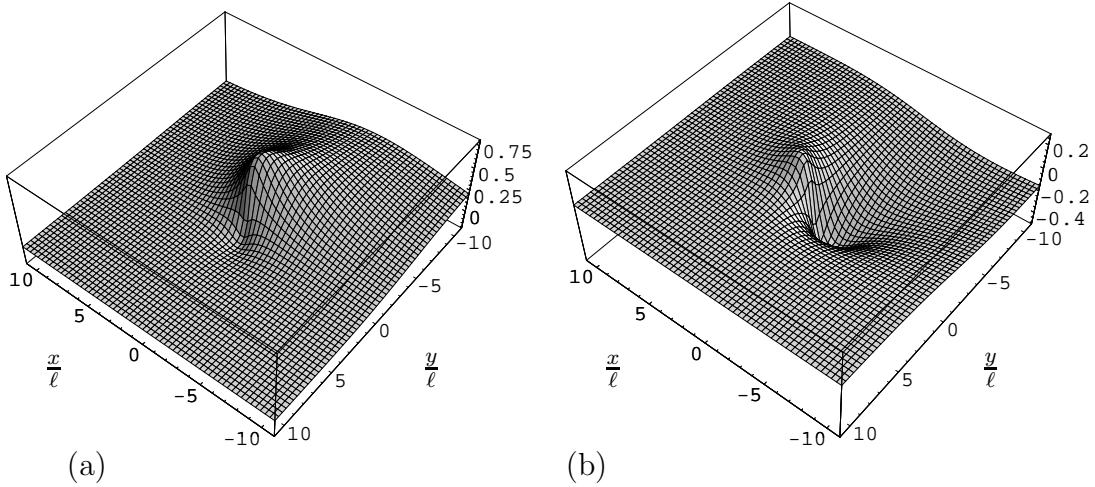


Figure 2: Three-dimensional plots of the elastic distortion of a screw dislocation in FGMs in units of $b/[2\pi]$ ($a_1 = 0.1/\ell$ and $a_2 = 0.2/\ell$): (a) β_{zx} and (b) β_{zy} .

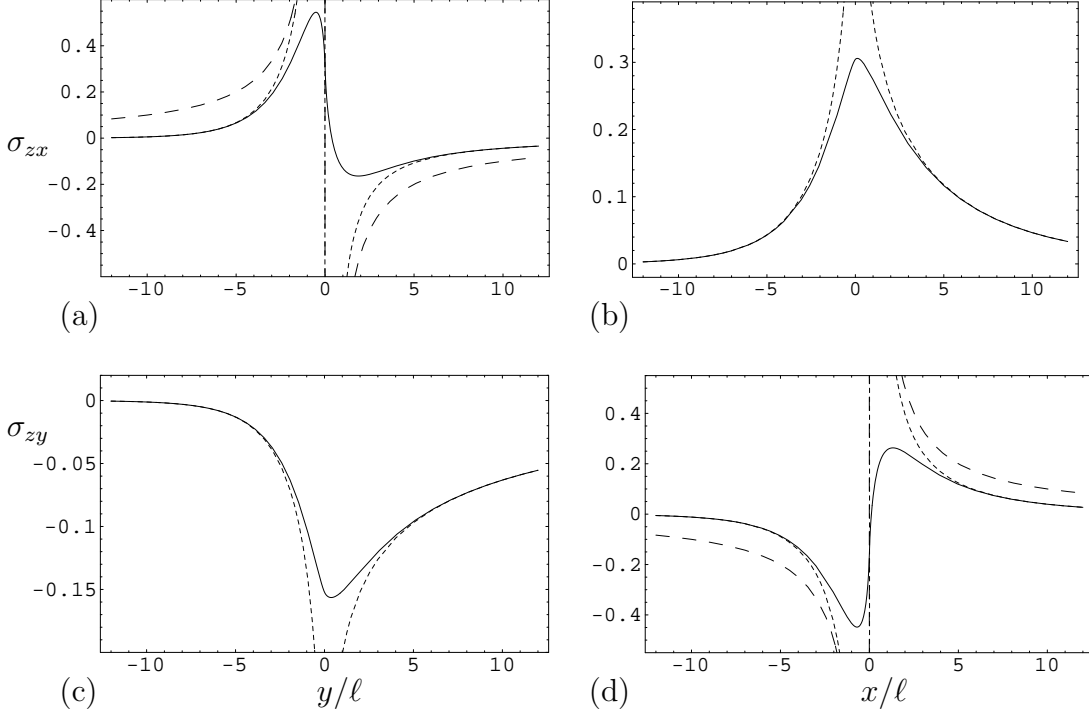


Figure 3: Stress of a screw dislocation in FGM using gauge theory (solid line) and elasticity (small dashed line) are given in units of $(\mu + \gamma)_0 b / [2\pi]$ and with $a_1 = 0.1/\ell$ and $a_2 = 0.2/\ell$: (a) $\sigma_{zx}(0, y)$, (b) $\sigma_{zx}(x, 0)$, (c) $\sigma_{zy}(0, y)$, (d) $\sigma_{zy}(x, 0)$. The dashed curves represent the stress in classical elasticity of a homogeneous medium.

materials. Evidently, $\beta_{zx}(y, 0)$ has a minimum at $y = 0$ and is asymmetric with respect to the plane $y = 0$: $\beta_{zy}(0, y) \neq \beta_{zy}(0, -y)$. For FGMs the far-field of the curve of the elastic distortion β_{zx} is for negative y higher than the $-1/y$ behavior known from homogeneous materials (see Fig. 1a). Due to the gradation of the material the elastic distortion in FGMs decays faster than in homogeneous media.

Using (11), the stresses can be derived to be

$$\sigma_{zx} = -\frac{(\mu + \gamma)^0 b}{2\pi} e^{(a_1 x + a_2 y)} \left(\frac{y}{r} [aK_1(ar) - \kappa K_1(\kappa r)] - a_2 [K_0(ar) - K_0(\kappa r)] \right), \quad (58)$$

$$\sigma_{xz} = -\frac{(\mu - \gamma)^0 b}{2\pi} e^{(a_1 x + a_2 y)} \left(\frac{y}{r} [aK_1(ar) - \kappa K_1(\kappa r)] - a_2 [K_0(ar) - K_0(\kappa r)] \right), \quad (59)$$

$$\sigma_{zy} = \frac{(\mu + \gamma)^0 b}{2\pi} e^{(a_1 x + a_2 y)} \left(\frac{x}{r} [aK_1(ar) - \kappa K_1(\kappa r)] - a_1 [K_0(ar) - K_0(\kappa r)] \right), \quad (60)$$

$$\sigma_{yz} = \frac{(\mu - \gamma)^0 b}{2\pi} e^{(a_1 x + a_2 y)} \left(\frac{x}{r} [aK_1(ar) - \kappa K_1(\kappa r)] - a_1 [K_0(ar) - K_0(\kappa r)] \right). \quad (61)$$

The stresses σ_{zx} and σ_{zy} are plotted in Figs. (3) and (4). It is important to note that the stresses (58)–(61) do not possess any singularity. Due to the gradation, the components (58)–(61) have extremum values near the dislocation line and a finite value at the dislocation line. For FGMs the far-field of the curves of the stresses are less than the $1/r$ behavior known from homogeneous materials (see Fig. 3a and 3d).

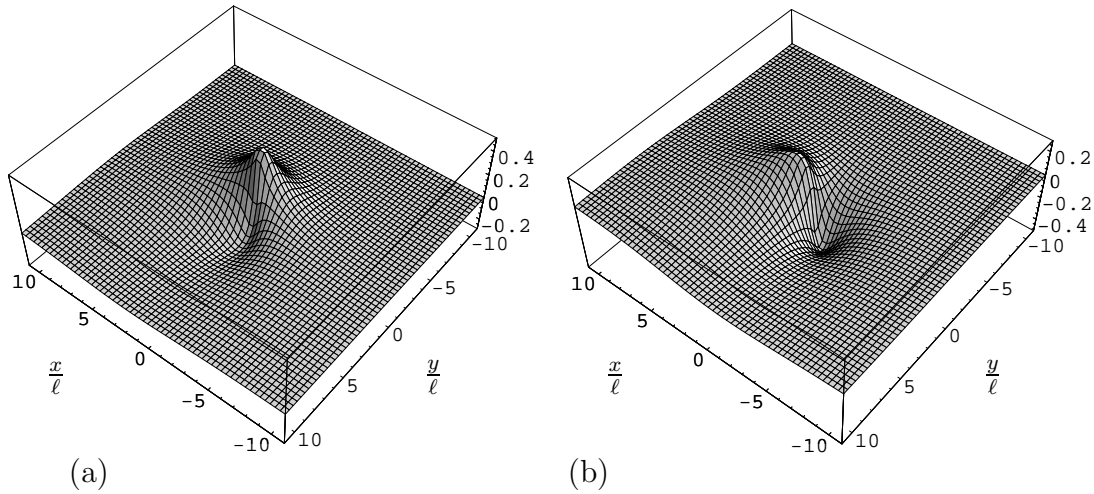


Figure 4: Three-dimensional plots of the stress of a screw dislocation in FGMs in units of $(\mu + \gamma)_0 b/[2\pi]$ ($a_1 = 0.1/\ell$, $a_2 = 0.2/\ell$): (a) σ_{zx} and (b) σ_{zy} .

We note that the expressions (56)–(61) coincide for $a_1 \rightarrow 0$ and $a_2 \rightarrow 0$ with the gauge-theoretical result obtained by Lazar and Anastassiadis (2009) for homogeneous materials and for $a_1 \rightarrow 0$ and $\gamma \rightarrow 0$ with the result given by Lazar (2007) using the strain gradient elasticity formulation for FGMs.

4 Conclusion

In this paper, we have investigated a straight screw dislocation in FGMs using the gauge theory of dislocations. In this framework, we have derived the field equations of dislocations for general nonhomogeneous media where the constitutive moduli depend on \mathbf{x} . Because of the gradation, the field equations contain gradients of the constitutive moduli. Subsequent we have specialized the field equations to exponentially graded materials. We found a new effective inverse length scale κ characteristic for the gauge-theoretical anti-plane problem of FGMs. We have calculated the elastic distortion, the force stress, the torsion and the pseudomoment stress fields produced by a screw dislocation in FGMs in the framework of dislocation gauge theory. All the analytical solutions are given in terms of the effective length scale $1/\kappa$ as well as the gradation length scale $1/a$. We found ‘perturbed’ Helmholtz equations as governing partial differential equations which we solved. We found exact analytical gauge-theoretical solutions which can be used by material scientists and design and manufacturing engineers. One hope is that these gauge solutions will provide new physical insight to possible improvements of designing FGMs. In the limit $\mathbf{a} \rightarrow 0$, we recover well-known results of a screw dislocation in the gauge theory for homogeneous media.

Acknowledgement

The author has been supported by an Emmy-Noether grant of the Deutsche Forschungsgemeinschaft (Grant No. La1974/1-3) and a Heisenberg fellowship (Grant No. La1974/2-1).

References

- Altan, B.C., Aifantis, E.C., 1997. On some aspects in the special theory of gradient elasticity. *J. Mech. Behav. Mater.* **8**, 231–282.
- Barnett, D.M., 1972. On the screw dislocation in an inhomogeneous elastic medium: The case of continuously varying elastic moduli. *Int. J. Solids Structures* **8**, 651–660.
- Chan, Y.-S., Gray, L.J., Kaplan, T., Paulino, G.H., 2004. Green’s function for a two-dimensional exponentially graded elastic medium. *Proc. R. Soc. Lond. A* **460**, 1689–1706.
- Chan, Y.-S., Paulino, G.H., Fannjing, A.C., 2006. Change of constitutive relations due to interaction between strain-gradient effect and material gradation. *Trans. ASME Journal of Applied Mechanics* **73**, 871–875.
- Chan, Y.-S., Paulino, G.H., Fannjing, A.C., 2008. Gradient elasticity for mode III fracture in functionally graded materials – part II: crack parallel to the material gradation. *Trans. ASME Journal of Applied Mechanics* **75**, 061015-1(11 pages).
- DeWit, R. 1973 Theory of disclinations IV, *J. Res. Nat. Bur. Stand. (U.S.)* **77A**, 607–658.
- Edelen, D.G.B., Lagoudas, D.C., 1988. *Gauge theory and defects in solids*, North-Holland, Amsterdam.
- Erdogan, F., 1995. Fracture mechanics of functionally graded materials. *Composites Eng.* **5**, 753–770.
- Eringen, A.C., 1999. *Microcontinuum field theories*. Springer, New York.
- Kadić, A., Edelen, D.G.B., 1983. A Gauge Theory of Dislocations and Disclinations, in: *Lecture Notes in Physics*, Vol. 174, Springer, Berlin.
- Markworth, A.J, Ramesh, K.S. Parks. Jr., W.P., 1995. Modelling Studies Applied to Functionally Graded Materials. *J. Mater. Sci.* **30**, 2183–2193.
- Lazar, M., 2002. An elastoplastic theory of dislocations as a physical field theory with torsion. *J. Phys. A: Math. Gen.* **35** (2002) 1983–2004.
- Lazar, M., Maugin, G.A. 2005. Nonsingular stress and strain fields of dislocations and disclinations in first strain gradient elasticity. *Int. J. Engng. Sci.* **43**, 1157–1184.
- Lazar, M., 2007. On the screw dislocation in a functionally graded material. *Mech. Res. Commun.* **34**, 305–311.
- Lazar, M., Anastassiadis, C., 2009. The gauge theory of dislocations: static solutions of screw and edge dislocations. *Phil. Mag.* **89** 199–231.
- Lazar, M., Hehl, F.W., 2010. Cartan’s spiral staircase in physics and, in particular, in the gauge theory of dislocations, *Foundations of Physics* **40**, 1298–1325.

- Martin, P.A., Richardson, J.D., Gray, L.J., Berger, J.R., 2002. On Green's function for a three-dimensional exponentially graded elastic solid. *Proc. R. Soc. Lond. A* **458**, 1931–1947.
- Mindlin, R.D., 1964. Micro-structure in linear elasticity. *Arch. Rat. Mech. Anal.* **16**, 51–78.
- Nowacki, W., 1986. *Theory of Asymmetric Elasticity*. PWN-Polish Scientific Publishers, Warszawa.
- Paulino, G.H., Fannjing, A.C., Chan, Y.-S., 2003. Gradient elasticity for mode III fracture in functionally graded materials– part I: crack perpendicular to the material gradation. *Trans. ASME Journal of Applied Mechanics* **70**, 531–542.
- Wang, X., Pan, E., 2008. On the screw dislocation in a functionally graded piezoelectric plane and half-plane. *Mech. Res. Commun.* **35**, 229–236.